NEMO split explicit free surface and tracer conservation

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1 Motivation

This note explores how NEMO time splitting implementation interferes with tracer conservation. A modification to the so called "forward" time splitting scheme (e.g. $ln\_bt\_fw = T$ in $namsplit$) is proposed to retrieve global and local conservation as in the unsplitted form of the equations proposed by [Leclair and Madec, 2009].

2 Background

NEMO time stepping is based on a three level time stepping scheme such that for any prognostic variable x:

$$x^{n+1} = x^{n-1} + 2\Delta t\ {F_x}^{n-1,\ n,\ n+1} \tag{1}$$

where ${F_x}^{n-1,\ n,\ n+1}$ is the time tendency computed at different time steps. As far as non-diffusive processes are concerned in the right-hand side of equation (1), these are given at central time step n, which translates into a leapfrog time stepping scheme (LF). To suppress the divergence of successive time steps inherent to the LF, a Robert-Asselin filter is applied:

$$x^{n*} = x^n + \gamma \left[x^{n+1} - 2x^n + x^{n-1*}\right] \tag{2}$$

where $\gamma = 0.001 - 0.1$ is a parameter (rn_atfp in the namelist).

3 Tracer conservation

The combination of LF and Robert-Asselin filtering, hereafter LF-RA, obscures somehow the overall conservation of tracers. Still, [Leclair and Madec, 2009] elegantly demonstrated that both global and local conservation of tracer can be achieved provided:

(i) A non-linear free surface method is used (e.g. time varying sea level anomalies are lumped into the model actual volume).

(ii) Robert-Asselin filtering is applied separately to thickness weighted tracer and cell thicknesses

(iii) Atmospheric forcing is defined at mid-steps interval (e.g. at steps $n + \frac{1}{2}$)

(iv) Corrective terms related to atmospheric forcing are added in equation (2).

Neglecting the forcing terms for a while, this lead to the following set of equations for both tracers and the free surface η:

$$\begin{aligned} h^{n+1}T^{n+1} &= h^{n-1*}T^{n-1*} + 2\Delta t\ \nabla \cdot \left[{F^n}_{adv} + {F^{n-1,\ n+1}}_{dif}\right] \\ h^{n*}T^{n*} &= h^nT^n + \gamma \left[h^{n+1}T^{n+1} - 2h^nT^n + h^{n-1*}T^{n-1*}\right] \\ \eta^{n+1} &= \eta^{n-1*} - 2\Delta t\ \nabla_h \cdot \overline{U}^n \\ \eta^{n*} &= \eta^n + \gamma \left[\eta^{n+1} - 2\eta^n + \eta^{n-1*}\right] \end{aligned} \tag{3}$$

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Here it is assumed that tracer advective fluxes are built from the same barotropic fluxes ($\overline{U}^n$) as used in the sea level time stepping. Vertical thicknesses are simply updated without any loss of generality such that:

$$h^{n+1} = h^{n-1*} + h^n \frac{\left(\eta^{n+1} - \eta^{n-1*}\right)}{H + \eta^n} + 2\Delta t F_h \tag{4}$$

The vertical sum of last term in (4) is null. It accounts for arbitrary displacements of the vertical coordinate (such as with the $\tilde{z}$ vertical coordinate of [Leclair and Madec, 2011]).

4 Interaction with time splitting

The work of [Leclair and Madec, 2009] is complemented here owing to the different implementation of the time splitting in the current code. At the time of the work of [Leclair and Madec, 2009], two different (possibly diverging) estimates of free surface were used. One issued from the barotropic loop (not used outside the barotropic integration, only the final, time averaged barotropic velocity was) and one stepped at the baroclinic level to update thicknesses. As a matter of fact, there wasn't much differences concerning conservation issues in splitted or unsplitted cases. A different implementation inspired from [Shchepetkin and McWilliams, 2005] has been implemented since then. The consequences relative to [Leclair and Madec, 2009] work are discussed below.

4.1 Centred integration: $ln\_bt\_fw = F$

Although not widely used in existing LF based models, the preferable way to implement time splitting should be between steps $n-1$ and $n+1$ (in fact eventually a little bit further than step $n+1$ in order to provide a time average of sea level and barotropic velocities, figure 1b). This naturally makes baroclinic to barotropic forcing terms centred in the middle of the integration window and coincident with the baroclinic integration ([Marsaleix et al., 2008]). Following [Shchepetkin and McWilliams, 2005], barotropic fluxes are then constructed thanks to a proper set of filtering weights such that:

$$\langle \eta_b \rangle^{n+1} = \langle \eta_b \rangle^{n-1} - 2\Delta t \; \nabla_h \cdot \langle\langle \overline{U}_b \rangle\rangle^n \tag{5}$$

Following again notations of [Shchepetkin and McWilliams, 2005], $\langle\ \rangle$ and $\langle\langle\ \rangle\rangle$ are primary and secondary sets of filtering weights, the latter being uniquely defined from the former. Subscripts refer to variables issued from the barotropic loop. Removing time averaging, i.e. setting $\langle\ \rangle$ to a Dirac function, one recovers a simple boxcar averaging of barotropic fluxes between $n-1$ and $n+1$ time steps. To avoid aliasing of fast barotropic motions (and remove instabilities induced by unperfect mode splitting), time averaging is however necessary[1]. The barotropic fluxes used in the tracer advection are then corrected by the fluxes entering the divergence term in equation (5). Since time averaged barotropic values also follow a leapfrog like time stepping, Asselin filtering is also applied which leads to following set of equations:

$$\begin{aligned} \langle \eta_b \rangle^{n+1} &= \langle \eta_b \rangle^{n-1*} - 2\Delta t \; \nabla_h \cdot \langle\langle \overline{U}_b \rangle\rangle^n \\ \langle \eta_b \rangle^{n*} &= \langle \eta_b \rangle^{n} + \gamma \left[\langle \eta_b \rangle^{n+1} - 2\langle \eta_b \rangle^{n} + \langle \eta_b \rangle^{n-1*} \right] \end{aligned} \tag{6}$$

The overall system of equations is completely equivalent to (3), sea level being here replaced by time averaged values issued from the barotropic loop. Thus, the demonstration of [Leclair and Madec, 2009] for global and local tracer conservation is here as valid as in the unsplitted case.

4.2 Forward integration: $ln\_bt\_fw = T$

The most commonly used implementation of time splitting with a leapfrog time stepping is between steps n and $n+1$ (Figure 1a). One can find at least three reasons to prefer this method to the centred integration:

(i) This is advantageous on a computational point of view: the number of iterations is roughly divided by a factor two[2].

[1] Adding ad hoc diffusion during barotropic integration, whether in the spatial or the temporal domain, can avoid this.

[2] It actually depends on the time span of the chosen filter. For a standard filtering window over a baroclinic time step the ratio is in fact 3/5

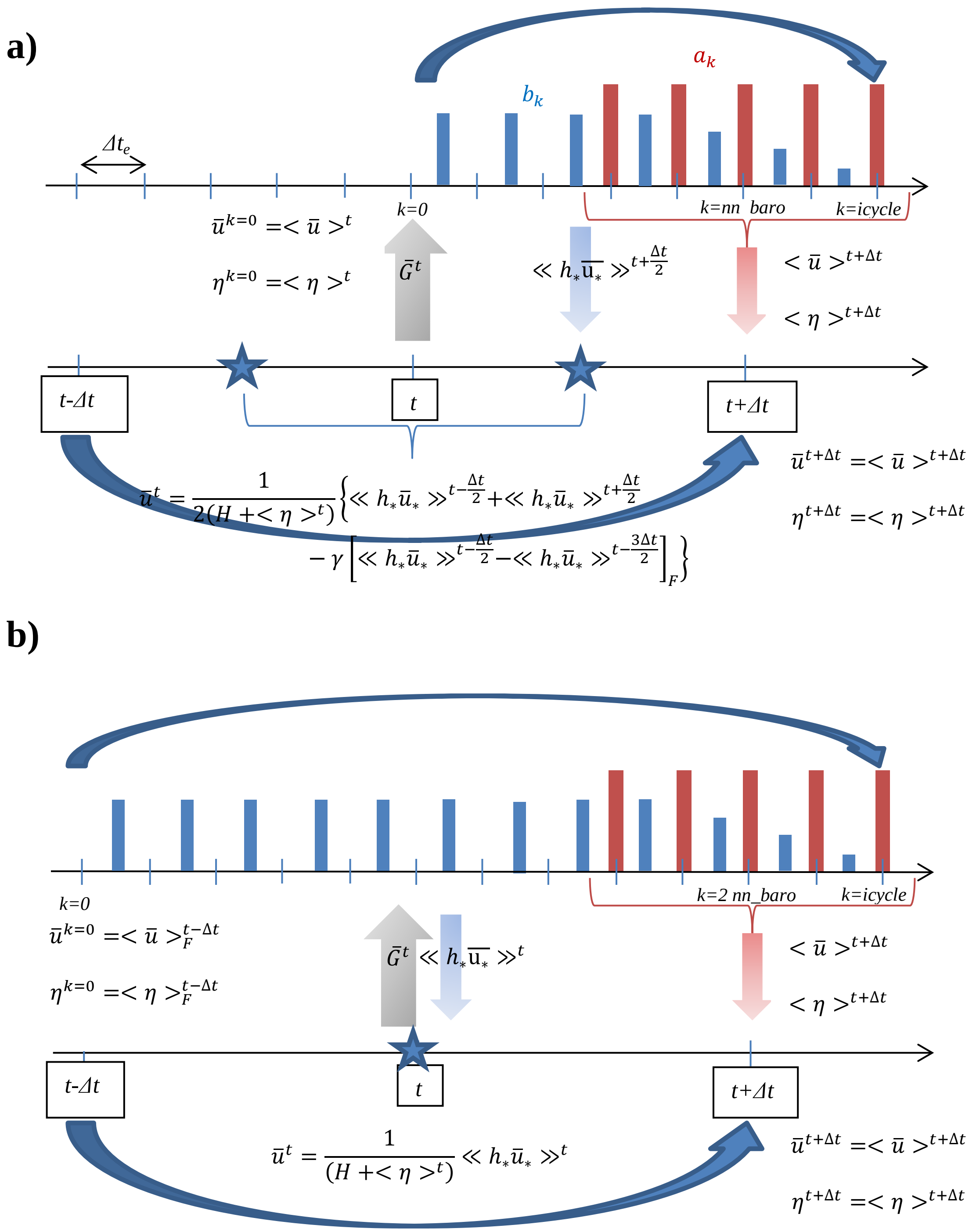

Figure 1: a) Forward and b) centred time splitting integrations with a leapfrog time stepping.

(ii) If time filtering on barotropic variables is not used (which removes a substantial damping on the barotropic solution itself), it is a bit unclear how to prevent diverging barotropic modes with the centred formulation ([Marsaleix et al., 2008]).

(iii) Such a scheme provides barotropic fluxes at half step intervals which is interesting to ensure volume conservation for on-line nesting applications (nested grids with refined time steps are usually advanced in a forward way. AGRIF software manages the time stepping that way). This argument is also beneficial to advect passive tracers online with a larger time step.

In that case, one obtain the following relationship issued from a single barotropic loop:

$$\langle\eta_b\rangle^{n+1} = \langle\eta_b\rangle^{n} - \Delta t\ \nabla_h \cdot \langle\langle\overline{U}_b\rangle\rangle^{n+\frac{1}{2}} \tag{7}$$

which expanded over two consecutive baroclinic time steps provides the corresponding barotropic fluxes to be used to advect tracers:

$$\langle\eta_b\rangle^{n+1} = \langle\eta_b\rangle^{n-1} - \Delta t\ \nabla_h \cdot \left[\langle\langle\overline{U}_b\rangle\rangle^{n-\frac{1}{2}} + \langle\langle\overline{U}_b\rangle\rangle^{n+\frac{1}{2}}\right] \tag{8}$$

The fundamental difference is that filtered sea level does not appear anymore in (8) compared to (6). One can still enforce a similar relationship between two barolinic steps, but at the expense of adding a source term in the barotropic equations, potentially unbalancing the system. This particular point breaks the demonstration issued from [Leclair and Madec, 2009] using the unsplitted form of the equations (3).

From this, one can either choose between maintaining global tracer conservation or constancy but not both. The first property is automatically obtained by still considering thickness weighted asselin filtering with unfiltered thicknesses:

$$h^n T^{n*} = h^n T^n + \gamma \left[h^{n+1}T^{n+1} - 2h^n T^n + h^{n-1}T^{n-1*}\right] \tag{9}$$

Constancy preservation can be achieved by compressing each cells accordingly:

$$\begin{aligned}(h^n + \gamma \left[h^{n+1} - 2h^n + h^{n-1}\right])T^{n*} &= \\ h^n T^n + \gamma \left[h^{n+1}T^{n+1} - 2h^n T^n + h^{n-1}T^{n-1*}\right]&\end{aligned} \tag{10}$$

The current NEMO release (v3.6) uses the second relationship. This is rather the result of assuming the same code during the filtering stage of tracers, whatever the time splitting formulation is, than a physically motivated choice. Well, that's a bug.

4.3 Maintaining conservation properties with forward time splitting

The idea is to recover the unsplitted system (3) by making a pseudo filtered sea level appear in equation (8). More precisely, we seek for corrective barotropic fluxes such that:

$$\langle\eta_b\rangle^{n+1} = \langle\eta_b\rangle^{n-1*} - \Delta t\ \nabla_h \cdot \left[\langle\langle\overline{U}_b\rangle\rangle^{n-\frac{1}{2}} + \langle\langle\overline{U}_b\rangle\rangle^{n+\frac{1}{2}} + \overline{U}_{cor}\right] \tag{11}$$

$\overline{U}_{cor}$ is only used in the advection of tracers and the continuity equation, equations (7) and (8) remain true.

Starting from equation (8), inserting the filtered free surface, one obtains:

$$\begin{aligned}\langle\eta_b\rangle^{n+1} = \langle\eta_b\rangle^{n-1*} &- \gamma\ \left(\langle\eta_b\rangle^{n} - \langle\eta_b\rangle^{n-1}\right)\\ &+ \gamma\ \left(\langle\eta_b\rangle^{n-1} - \langle\eta_b\rangle^{n-2}\right)\\ &- \gamma^2 \left(\langle\eta_b\rangle^{n-1} - \langle\eta_b\rangle^{n-2}\right)\\ &+ \gamma^2 \left(\langle\eta_b\rangle^{n-2} - \langle\eta_b\rangle^{n-3}\right)\\ &...\\ &- \Delta t\ \nabla_h \cdot [...]\end{aligned} \tag{12}$$

which can be arranged taking into account equation (7) into:

$$\begin{aligned}\langle\eta_b\rangle^{n+1} = \langle\eta_b\rangle^{n-1*} &- \Delta t\ \nabla_h \cdot \left[\langle\langle\overline{U}_b\rangle\rangle^{n-\frac{1}{2}} + \langle\langle\overline{U}_b\rangle\rangle^{n+\frac{1}{2}}\right]\\ &- \Delta t\ \nabla_h \cdot \left[-\gamma\ \left(\langle\langle\overline{U}_b\rangle\rangle^{n+\frac{1}{2}} - \langle\langle\overline{U}_b\rangle\rangle^{n-\frac{3}{2}}\right)\right]\\ &- \Delta t\ \nabla_h \cdot \left[-\gamma^2 \left(\langle\langle\overline{U}_b\rangle\rangle^{n-\frac{3}{2}} - \langle\langle\overline{U}_b\rangle\rangle^{n-\frac{5}{2}}\right)\right]\\ &- \Delta t\ \nabla_h \cdot [...]\end{aligned} \tag{13}$$

Neglecting small terms in the filtering, i.e. those involving γ^p with $p > 1$, the corrective term has a similar form as the one proposed by [Leclair and Madec, 2009] except that here it is not related to atmospheric forcing but to the difference of the local divergence between two consecutive time steps. Without any approximation (13) can be expressed as the following recursive relationship:

$$\begin{aligned}\langle\eta_b\rangle^{n+1} &= \langle\eta_b\rangle^{n-1*} - \Delta t\ \nabla_h \cdot \left[\langle\langle\overline{U}_b\rangle\rangle^{n-\frac{1}{2}} + \langle\langle\overline{U}_b\rangle\rangle^{n+\frac{1}{2}} - \gamma\overline{U}_-^{n-1*}\right] \\ \overline{U}_-^{n*} &= \overline{U}_-^{n-1*} + \gamma\left[\langle\langle\overline{U}_b\rangle\rangle^{n+\frac{1}{2}} - \langle\langle\overline{U}_b\rangle\rangle^{n-\frac{1}{2}}\right]\end{aligned} \tag{14}$$

We assumed that time stepping sequence begins with a forward time step. Or assuming (8) is still valid, this is equivalent to:

$$\begin{aligned}\langle\eta_b\rangle^{n-1*} &= \langle\eta_b\rangle^{n-1} - \gamma\Delta t\ \nabla_h \cdot \left[\overline{U}_-^{n-1*}\right] \\ \overline{U}_-^{n*} &= \overline{U}_-^{n-1*} + \gamma\left[\langle\langle\overline{U}_b\rangle\rangle^{n+\frac{1}{2}} - \langle\langle\overline{U}_b\rangle\rangle^{n-\frac{1}{2}}\right]\end{aligned} \tag{15}$$

The recurrence for (15) can be easily demonstrated. The system of equations is now identical to (3), so that the demonstration for global conservation also holds. Taking a constant tracer value in its prognostic equation, leads to the ”true” sea level equation (8) ensuring local conservation.

4.4 Practical changes between the two formulations

From the preceding discussion there should be no significant differences between the two time splitting options in the code to ensure tracer conservation. In both cases, asselin filtering is applied identically on thicknesses. The barotropic fluxes used for tracer advection (and in the continuity equation) in the ”forward” case are however different as commonly assumed: they are given by the recursive relationship (14) which implies saving two additional 2d arrays.

Forcing terms, and in particular surface water fluxes, have been neglected so far. Since they are defined at mid-steps interval, taking the appropriate value during the barotropic integration is straightforward. Corrections proposed by [Leclair and Madec, 2009] are therefore, and whatever the time splitting scheme is, as valid as in the unsplitted case. The proposed modifications have been validated in the ORCA2 global configuration (see Figure 2). All free surface methods garanty exact local and global tracer conservation.

References

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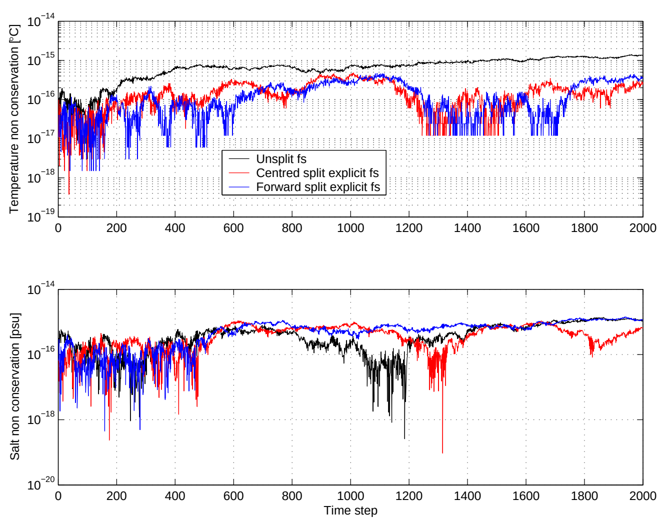

Figure 2: Log plot of heat a) and salt b) global non conservation in ORCA2 model depending on the free surface method.